HIGHER SPIN VERTEX MODELS WITH DOMAIN WALL BOUNDARY CONDITIONS

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Abstract. We derive determinant expressions for the partition functions of spin-$k/2$ vertex models on a finite square lattice with domain wall boundary conditions.

0. Introduction

In [1], Korepin introduced the concept of domain wall (DW) boundary conditions for the six vertex (or spin-1/2) model on a finite lattice, and proposed recursion relations that fully determine the partition function in that case. In [2], Izergin obtained a determinant solution of Korepin's recursion relations. In this work, inspired by Slavnov's inner product formula for higher spin models [3], we derive determinant expressions for the spin-$k/2$ DW partition functions[1], $k \in \boldsymbol{N}$, using the fact that these models are related to the spin-1/2 model using fusion [4, 5].

Basically, we show that fusion can be applied at the level of spin-1/2 partition functions to obtain spin-$k/2$ partition functions, for any $k \in \boldsymbol{N}$. More specifically, our result, in words, is that appropriate specializations of the rapidity variables in Izergin's determinant expression for the spin-1/2 partition function on a $kL \times kL$ lattice (followed by suitable normalizations) yield determinant expressions for the spin-$k/2$ partition functions on an $L \times L$ lattice.

In sections 1 and 2, we briefly introduce the spin-$k/2$ vertex models, and outline the fusion procedure. In 3 and 4, we motivate our result, then outline its proof. In 5, we give the spin-1 partition function as a specific example of our general result, that has the benefit of allowing for an independent Izergin-type proof (which is not available for higher spin models). In 6, we derive the homogeneous limit of the spin-$k/2$ result, and comment on combinatorics of higher spin models. Finally, an appendix contains technical details. The presentation is elementary and (almost) self-contained.

1. Vertex models

Definitions related to vertex models. We work on a square lattice consisting of L horizontal lines (labelled from bottom to top), L vertical lines (labelled from left to right) and L^2 intersection points.

We assign the i-th horizontal line an orientation from left to right, and a complex rapidity variable x_i. We assign the j-th vertical line an orientation from bottom to top, and a complex rapidity variable y_j. All rapidity variables are independent, unless specifically indicated to be otherwise.

$\{\mathbf{x}\}$ is a set of rapidity variables $\{x_1, \cdots, x_C\}$, where the cardinality of the set should be clear from context.

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[1]In the sequel, *partition function* will refer to *DW partition function* unless otherwise indicated. For applications of the Izergin-Korepin determinant formula in statistical mechanics, see [6]. For applications to algebraic combinatorics, see [7].

$\{\mathbf{x}\}_i$ is a set $\{x_1, x_2, \cdots, x_C\}$, but with the x_i element missing.
A k-stack $\{\mathbf{x}|k\}$ is a set of k variables of the form $\{x, x+1, x+2, \cdots, x+k-1\}$.

A bond is a line segment between two intersection points.
A boundary, or extremal bond is a line segment, at the boundary of a line, attached to a single intersection point.

In a spin-$k/2$ models, we assign each bond κ arrows, where $\kappa \in \{k, k-2, \cdots, k \bmod 2\}$. All κ arrows, on the same bond, point in the same direction.

The κ arrows on a bond define a *spin* variable on that bond.
The magnitude of spin on a bond is $\kappa/2$.
The sign of spin on a bond is positive (negative) if the κ arrows are oriented in the same (opposite) direction as (to) the rapidity flow in that bond.

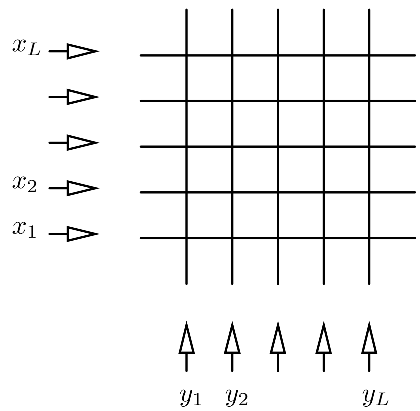


Figure 1. *A finite square lattice, with oriented lines and rapidities.*

A vertex v_{ij}, is the intersection point of the i-th horizontal line and the j-th vertical line, together with the 4 bonds attached to it, and the arrows on them.
a weight, w_{ij}, is a function assigned to a vertex, v_{ij}, that depends on the difference of rapidity variables flowing through that vertex. In exactly solvable models, the weights satisfy the Yang Baxter equations [8].

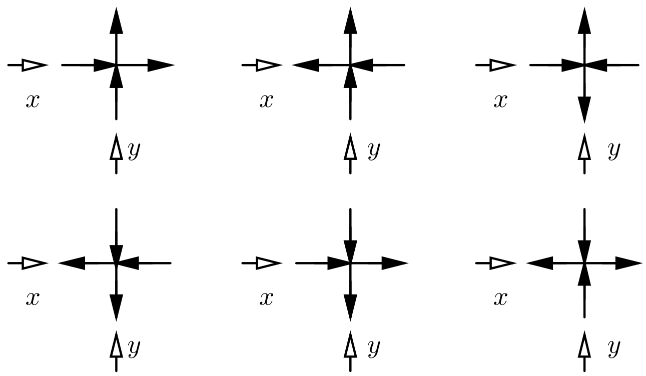
Frequently used abbreviations. We frequently use the bracket notation $[\mathbf{x}] = \sinh(\lambda x)$ (where λ is a constant, *'crossing'* parameter that characterizes the model) and the related product notation $[\mathbf{x}]_m = [x][x-1]\cdots[x-m+1]$. We also use the abbreviations $\tilde{x} = x+1$ and $u_{ij} = -x_i + y_j$.
The partition function of a spin-$k/2$ vertex model, on an $L \times L$ lattice, $Z^{k/2 \times k/2}_{L \times L}$, is a weighted sum over all configurations, that satisfy certain boundary conditions. The weight of a configuration is the product of the weights, w_{ij}, of the vertices v_{ij}.

$$Z^{k/2 \times k/2}_{L \times L}\Big(\{\mathbf{x}\}, \{\mathbf{y}\}\Big) = \sum \prod_{\text{vertices}} w_{ij} \tag{1}$$

Conservation of spin flow. The vertex models, that we discuss in this work, conserve spin flow: In all vertices that have non-zero weight, the net incoming spin (measured with respect to rapidity inflow) equals the net outgoing spin (measured with respect to rapidity outflow).

The spin-$1/2$ model. There are six vertex types that conserve spin flow in the spin-1/2 model. They are shown in figure 2 below. For convenience, we label the vertices by their weights: an a vertex has weight $a(x, y)$, and so forth.

We do not need to distinguish vertices that share the same weight, except in one case (the $c+$ vertex) mentioned below. The six vertices of the spin-1/2 and their weights are shown in figure 2. The weights of every two vertices in the same column are equal and shown below them.



$a(x,y) = [u+1]$ $\quad$ $b(x,y) = [u]$ $\quad$ $c(x,y) = [1]$

Figure 2. *The six vertices of the spin-*$1/2$ *model and their weights.* $u = -x + y$.

The $c+$ vertex. The vertex with all arrows pointing inwards from left and right, and all arrows pointing outwards from above and below, plays a rather special role in this work. We refer to it as the $c+$ vertex. There is a unique $c+$ vertex in every spin-$k/2$ model. The spin-2 $c+$ vertex is shown in figure 3.

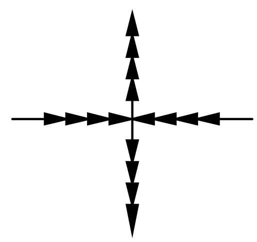


Figure 3. *The spin-2* $c+$ *vertex.*

Domain wall (DW) boundary conditions. Consider the spin-1/2 model on a finite square lattice, and require that the boundary arrows have the same orientation as the arrow on the corresponding boundary of the $c+$ vertex: all arrows on the left and right boundaries point inwards, and all arrows on the upper and lower boundaries point outwards. The $c+$ vertex is a DW configuration on a 1×1 lattice.

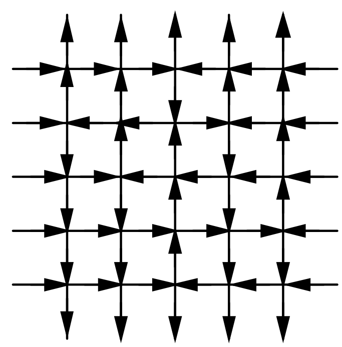


Figure 4. *A DW configuration.*

The Izergin-Korepin (IK) determinant expression for the spin-1/2 partition function [1, 2] is

$$Z^{1/2 \times 1/2}_{L \times L}\Big(\{\mathbf{x}\}, \{\mathbf{y}\}\Big) = \frac{\prod_{i,j=1}^{L} [-x_i + y_j + 1]_2}{\prod_{1 \le i < j \le L} [-x_i + x_j][-y_j + y_i]} det\Big(M^{1/2 \times 1/2}_{L \times L}\Big) \tag{2}$$

where

$$M^{1/2 \times 1/2}_{L \times L, ij} = \frac{[1]}{[-x_i + y_j + 1]_2}$$

Definitions related to partition functions, matrices and normalizations.
$Z^{k/2\times k/2}_{L\times L}$ is the partition function of the spin-$k/2$ model on an $L\times L$ lattice.
$M^{k/2\times k/2}_{L\times L}$ is the $L\times L$ IK matrix, but with a $k\times k$ block structure, as will be explained below.
$M^{k/2\times k/2}_{kL\times kL,ij}$ is the $k\times k$ ij-th block of $M^{k/2\times k/2}_{kL\times kL}$. It depends on the rapidities $\{x_i, y_j\}$.
$B^{k\times k}_{ij}$ is the ij-th $k\times k$ block of the lattice.
$N^{k/2\times k/2}_{kL\times kL}$ is the normalization function of $M^{k/2\times k/2}_{L\times L}$ that sets the weights of the spin-$k/2$ vertices to $[k]_k$.

Spin-$k/2$ models. We will not need the weights of spin-$k/2$ model in all generality. They can be deduced from those of the fused elliptic height models, [5], as follows.

Spin-$k/2$ vertex weights from elliptic height weights.

§1. Take the trigonometric limit: set the elliptic nome, which appears in the weights of elliptic models, $q\to 0$. This reduces the weights to ratios of products of trigonometric functions (for pure imaginary values of the crossing parameter).

§2. Take the vertex limit: set the height shift parameter, that appears in the weights of height models, $\zeta\to\pm\infty$. This eliminates dependence on the height variables.

§3. Symmetrize the resulting weights so that the weights of the c-type vertices are equal.

2. Fusion

As the spin-$k/2$ vertices are obtained from the spin-1/2 vertices using fusion, we wish to recall how fusion works, following [5][2].

Definitions related to boundaries. A boundary of length L is a set of L parallel extremal bonds. A vertical boundary consists of horizontal extremal bonds. A horizontal boundary consists of vertical extremal bonds. Notice that, in our definition, a boundary cannot be closed. A closed boundary will consist of a sequence of horizontal and vertical boundaries.
A σ-configuration is an arrangement of spins on a boundary, with total spin σ. A boundary of length L, in a spin-$k/2$ model, has $\sigma\in\{kL/2, kL/2-1,\cdots,-kL/2\}$.
A σ-set is the set of all σ-configurations on a boundary.
A σ-representative of a σ-set, is any single uniquely defined configuration in that set, that we select to represent the entire set. In this work, we choose that to be the configuration with spins ordered in terms of their values, with larger (more positive) spins to the left of (lower than) smaller (less positive) spins in the case of horizontal (vertical) boundaries.

An inflow (outflow) boundary is one that rapidity variables flow into (out of), as seen from the inside of the region that it bounds.

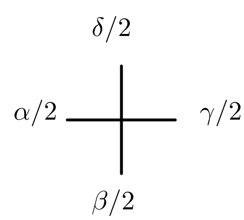


Figure 5. *Spin values of a general vertex in a spin-$k/2$ model.*

[2]Although [5] studies fusion in the context of elliptic height models, it is convenient to start from there, as their exposition is explicit.

Fusion procedure. Following [5], to compute the weight of the generic spin-$k/2$ vertex shown in figure 5, where $(\alpha, \beta, \gamma, \delta) \in \{k, k-1, \cdots, -k\}$ and $\alpha+\beta = \gamma+\delta$, we start from the set of all spin-1/2 configurations on a $k \times k$ lattice, with boundaries that match in total spin values those of the vertex that we wish to produce (the right boundary has total spin $\alpha/2$, *etc*), and proceed as follows.

§1. Set $\{x_1, x_2, \cdots, x_k\}$ to $\{\mathbf{x_1}|k\}$, and set $\{y_1, y_2, \cdots, y_k\}$ to $\{\mathbf{y_1}|k\}$. x_1 and y_1 will be the rapidities of the resulting spin-$k/2$ vertex.
§2. Sum over all $\alpha/2$-configurations on the left (inflow) boundary.
§3. Sum over all $\beta/2$-configurations on the lower (inflow) boundary.
§4. Take the $\gamma/2$-configuration on the right (outflow) boundary to be the unique $\gamma/2$-representative. No summation over configurations is performed.
§5. Take the $\delta/2$-configuration on the upper (outflow) boundary to be the unique $\delta/2$-representative. No summation over configurations is performed.
§6. Normalize the result[3], by dividing with

$$N_{1\times 1}^{k/2\times k/2}(x_1, y_1) = \prod_{p=0}^{k-1} [-x_1 + y_1 + p]_{k-1} \tag{3}$$

Remarks. To perform fusion, in our convention, inflow boundaries are summed over, while outflow boundaries are set to representative configurations. Further, we could have set the outflow boundaries to any σ-configuration with the correct net spin σ. However, using the Yang-Baxter equations, one can show that the result is independent of the choice [5].

3. Spin-$k/2$ partition functions: Motivation

Suppose we wish to obtain the spin-2 $c+$ vertex of figure 3. Following the fusion procedure, we need to consider the 4×4 DW spin-1/2 partition function, shown in figure 3.

But this case is very simple: Because of the boundary conditions, all σ-sets have exactly one configuration each, and we do not need to sum over left and lower configurations, or choose any right and upper ones. All we need to do is to set the rapidities to the right values, and normalize suitably.

Now, suppose we do not wish to fuse all the way down to the $c+$ vertex, which is, the 1×1 DW spin-2 partition function, but only *half way* to the 2×2 DW spin-1 partition function.

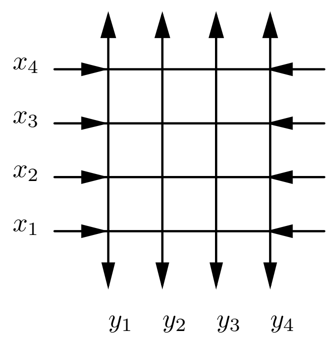


Figure 6. 4×4 *spin*-1/2 *partition function.*

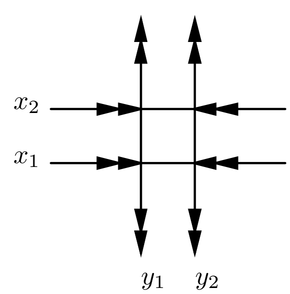


Figure 7. 2×2 *Spin-1 partition function.*

[3]Our normalization is different from that of [5]. We choose to normalize the $c+$ vertex of the spin-$k/2$ model to $[k]_k$. In [5], the $c+$ vertices are normalized to 1 (up to phases), in the trigonometric vertex limit that we are interested in.

Partial fusion. It seems plausible that all we need to do in this case is to 2-stack the rapidity variables, and normalize suitably. In other words, we need *partial fusion* as follows.

§1. Consider the spin-1/2 model on a $kL \times kL$ lattice with DW boundary conditions.

§2. Set $\{x_1, x_2, \cdots, x_{kL}\}$ to L k-stacks $\{\{\mathbf{x_1}|k\}, \{\mathbf{x_2}|k\}, \cdots, \{\mathbf{x_L}|k\}\}$ and $\{y_1, y_2, \cdots, y_{kL}\}$ to L k-stacks $\{\{\mathbf{y_1}|k\}, \{\mathbf{y_2}|k\}, \cdots, \{\mathbf{y_L}|k\}\}$. Under this restriction of variables, the $M^{1/2\times 1/2}_{kL\times kL}$ IK matrix, that we started with, is now denoted by $M^{1/2\times 1/2}_{kL\times kL}$.

§3. Normalize so that the weight of the spin-$k/2$ $c+$ vertex is a constant[4] by dividing with

$$N^{k/2\times k/2}_{L\times L}\Big(\{\mathbf{x}\},\{\mathbf{y}\}\Big) = \prod_{1\le i,j\le L}\prod_{p=0}^{k-1}[-x_i+y_j+p]_{k-1}$$

Following the above procedure, we obtain the following expression for the spin-$k/2$ partition function

$$Z^{k/2\times k/2}_{L\times L}\Big(\{\mathbf{x}\},\{\mathbf{y}\}\Big) = \frac{\prod\limits_{1\le i,j\le L}\prod\limits_{p=1}^{k}[-x_i+y_j+p]_{k+1}}{\prod\limits_{1\le i<j\le L}\prod\limits_{p=0}^{k-1}[-x_i+x_j+p]_k\prod\limits_{p=0}^{k-1}[-y_j+y_i+p]_k} \times det\Big(M^{k/2\times k/2}_{kL\times kL}\Big) \tag{4}$$

Equation 4 is our main result. In the next section, we show that partial fusion, as outlined above, works, and leads to spin-$k/2$ partition functions, with *no missing or unwanted* configurations. Technical details of how equation 4 is obtained are in the appendix.

4. Spin-$k/2$ partition functions: Proof

We wish to show that, starting from a weighted sum over spin-1/2 configurations, on a $kL \times kL$ lattice, and dividing the lattice into L^2 $k \times k$ blocks, we can fuse these blocks one by one, and obtain a weighted sum over spin-$k/2$ configurations with the correct spin-$k/2$ weights.

In particular, we also wish to show that This *partial* fusion procedure leads to *all* required spin-$k/2$ configurations, *and no more*. In fact, it will turn out that this procedure is bijective in the sense that every step is reversible.

The following is an outline of the proof, together with a simple running example.

4.1. **Outline of proof.**

§1. Consider a DW spin-1/2 model on a $kL \times kL$ lattice. As an example, we take $k = 2$ and $L = 3$.

[4]The rationale of normalization is to put the result in a practical, recognizable form, and in particular to avoid that the weights have common, spurious poles or zeros.

§2. Set the horizontal rapidities into L k-stacks of the form $\{\{\mathbf{x_1}|k\}, \{\mathbf{x_2}|k\}, \cdots \{\mathbf{x_L}|k\}\}$, and similarly for the vertical rapidities. In our example, we obtain $\{\{x_1|2\},\{x_2|2\},\{x_3|2\}\}$ and $\{\{x_1|2\},\{x_2|2\},\{x_3|2\}\}$

§3. Divide the lattice into L^2 $k \times k$ blocks, $B_{ij}^{k\times k}$, where i is the block row index, j is the block column index, and $\{i, j\} \in \{1, 2, \cdots, L\}$. Each block has one independent horizontal rapidity x_i, and one independent vertical rapidity y_j. In our example, we obtain nine 2×2 blocks.

§4. Order the blocks, firstly in terms of row position: blocks with a smaller i precede those with a larger i, then in terms of column position: for equal i indices, blocks with a smaller j precede those with larger j. In our example, the order is $\{\ B_{11}^{2\times2},\ B_{12}^{2\times2},\ B_{13}^{2\times2},\ B_{21}^{2\times2},\ B_{22}^{2\times2},\ B_{23}^{2\times2},\ B_{31}^{2\times2},\ B_{32}^{2\times2},\ B_{33}^{2\times2}\ \}$

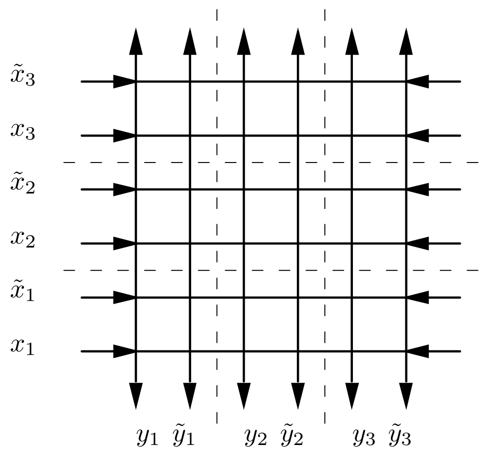


Figure 8. *Spin-1/2 partition function with 2-stacked rapidity variables.* $\tilde{z} = z + 1$.

§5. Consider the partition function $Z_{kL\times kL}^{1/2\times1/2}$ as a sum over products of two partition functions: that of $B_{11}^{k\times k}$, and that of the rest of the lattice, R_{11}, that is

$$Z_{kL\times kL}^{1/2\times1/2} = \sum_{\{h_1,v_1\}} B_{11}^{k\times k} R_{11} \tag{5}$$

where h_1 and v_1 stand for the horizontal and vertical common spin boundaries. The sum is over all configurations on the common boundaries of $B_{11}^{k\times k}$ and R_{11}.

§6. Each $B_{11}^{k\times k}$ term in the above sum is DW on an inflow boundary, and has a single σ-configuration on an outflow boundary. Since a DW boundary condition corresponds to a σ-set with a single σ-configuration, the inflow boundaries are (trivially) summed, while the outflow boundaries are fixed to a certain σ-configuration. But, from fusion, all such partition functions are equal to the one with σ-representatives on the outflow boundaries. This allows us to simplify the sum in the previous equation to

$$Z_{L\times L}^{1/2\times1/2} = \sum \left(B_{11}^{k\times k} \sum R_{11} \right) \tag{6}$$

where the first (right most) sum is over all σ-configurations, in an allowed σ-set on the inflow boundaries of R_{11}, $B_{11}^{k\times k}$ has σ-representatives on the outflow boundaries, and the second (left most) sum is over all σ-sets on the common boundaries. Notice that we need to be clear about what is being

summed over in each sum. This is because, before fusion, all elements in all allowed σ-sets are summed over, while, after fusion, only σ-representatives are summed over.

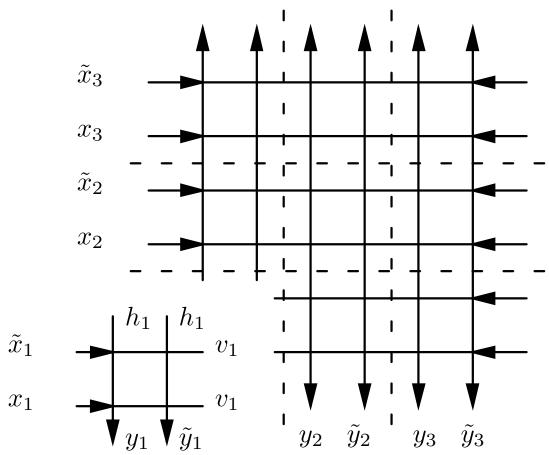


Figure 9. *Detaching a* 2×2 *spin*-$1/2$ *block.* $\tilde{z} = z + 1$.

§7. Using fusion to write $B_{11}^{k \times k}$ as a spin-$k/2$ vertex, we end up with a sum over products of a 1×1 spin-$k/2$ non-DW partition function and a spin-$1/2$ non-DW partition function (the original lattice minus $B_{11}^{k \times k}$).

$$Z_{kL \times kL}^{1/2 \times 1/2} = \sum \left(Z_{1 \times 1}^{k/2 \times k/2} \sum Z_{(kL \times kL)-(1 \times 1)}^{1/2 \times 1/2} \right) \tag{7}$$

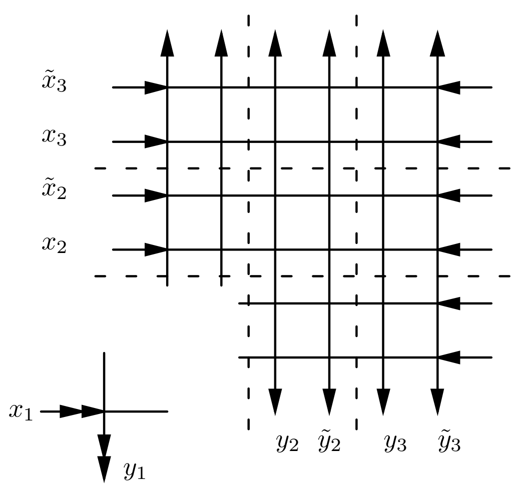


Figure 10. *Fusing a detached* 2×2 *spin*-$1/2$ *block into a* 1×1 *spin-1 partition function.* $\tilde{z} = z + 1$.

§8. Next, we consider the next ranking block in $Z_{(kL \times kL)-(1 \times 1)}^{1/2 \times 1/2}$, namely $B_{12}^{k \times k}$, and write

$$Z_{(kL \times kL)-(1 \times 1)}^{1/2 \times 1/2} = \sum B_{12}^{k \times k} Z_{(kL \times kL)-(2 \times 1)}^{1/2 \times 1/2} \tag{8}$$

Using the same reasoning, and notation, as above, we can write

$$Z^{1/2\times 1/2}_{(kL\times kL)-(1\times 1)} = \sum \left(B^{k\times k}_{12} \sum Z^{1/2\times 1/2}_{(kL\times kL)-(2\times 1)} \right) \tag{9}$$

Using fusion to re-write $B^{k\times k}_{12}$ as a spin-$k/2$ vertex, combining the above results, and summing over the common boundaries of the two 1×1 spin-$k/2$ partition functions, we can write the initial spin-1/2 non-DW partition function as a sum over products of two objects: a spin-$k/2$ non-DW partition function consisting of 2 vertices, and the remaining spin-1/2 lattice.

It should be clear from the above that we are fusing the initial spin-1/2 lattice, one block at a time, to a spin-$k/2$ lattice. It should also be clear that this can be done block by block, that the procedure is reversible, and that we end up with the desired $L \times L$ DW spin-$k/2$ model.

The DW boundary conditions of the final, spin-$k/2$ configurations follow from the DW boundary conditions of the initial spin-1/2 configurations.

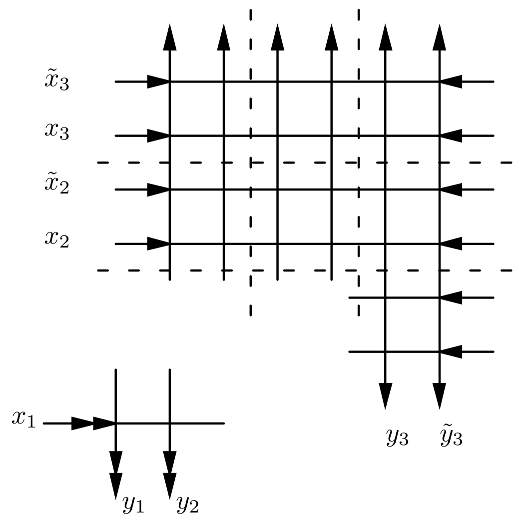


Figure 11. *Detaching, fusing the second* 2×2 *spin-*$1/2$ *block, fusing it to form a spin-1 vertex, then attaching the latter to the first spin-1 vertex.* $\tilde{z} = z + 1$.

This concludes our outline of the proof of equation 4, from which the reader can recover a formal proof if necessary.

5. Example: Spin-1 model

Consider the spin-1 (19-vertex, or Zamolodchikov-Fateev model [9]) model constructed by fusion [5]. The vertex weights, which can be computed explicitly using fusion, are listed (for example) in [10]. In the following, we list only those vertices that we need and their the weights. Vertices that are related to those listed, by inverting of all arrows, have the same weights.

[5]There are many 19-vertex models. The Zamolodchikov-Fateev model that we are considering here is only on of these. Two others are listed in [10].

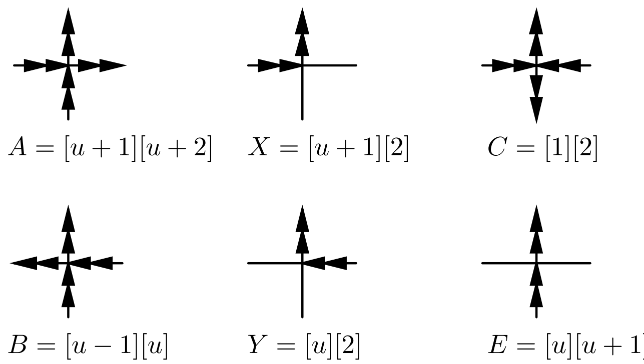


Figure 12. *A subset of the vertices of the spin-*1 *model and their weights.* $u = -x + y$.

Re-writing equation 4, for $k = 2$, in terms of vertex weights of the spin-1 model, we obtain the following expression for the spin-1 partition function

$$Z^{1\times 1}_{L\times L}\Big(\{\mathbf{x}\},\{\mathbf{y}\}\Big) = \frac{\prod_{1\le i,j\le L} A(-x_i+y_j)E(-x_i+y_j)B(-x_i+y_j)}{\prod_{i<j} E(-x_i+x_j)B(-x_i+x_j)E(y_i-y_j)B(y_i-y_j)} \times det\Big(M^{2\times 2}_{2L\times 2L}\Big) \tag{10}$$

$$\begin{aligned} M^{2\times2}_{2j-1,2i-1} &= 1/E(-x_i+y_j), \quad & M^{2\times2}_{2j-1,2i} &= 1/B(-x_i+y_j), \\ M^{2\times2}_{2j,\ \ 2i-1} &= 1/A(-x_i+y_j), \quad & M^{2\times2}_{2j,\ \ 2i} &= 1/E(-x_i+y_j) \end{aligned} \tag{11}$$

Independent check. The determinant expression of the spin-1 partition function, obtained above, allows for an independent check of our general result, in the sense that one can take the determinant expression as a conjecture, and show that it is correct.

Just as in the spin-1/2 case [1, 2], one needs to show that the LHS of equation 10 (the partition function) has certain properties that uniquely determine it completely, then show that the RHS (the proposed determinant expression) satisfies the same properties. An outline of the basic steps is as follows.

The LHS of equation 10. The following properties fully characterize the partition function

Symmetry Using the Yang Baxter equations, one can show that the partition function is a symmetric function in $\{\mathbf{x}\}$ and in $\{\mathbf{y}\}$.

Degree One can easily show that, on any extremal row or column, there is exactly one rapidity independent vertex (namely a $c+$ vertex), *or* two vertices that are trigonometric polynomials of degree 1 each, while all other vertices are of degree 2. This means that the partition function is a trigonometric polynomial of degree $2L - 2$ in the rapidity variable in that row or column, and by symmetry in every other variable.

Recursion relations Below, we will show that the partition function satisfies $2L$ recursion relations in each rapidity variable, which is more than we actually need.

The initial condition By construction, the 1×1 DW partition is a $c+$ vertex.

The RHS of equation 10. Next we show that the RHS satisfies the same properties as the LHS.

Symmetry By direct calculation, one can show that the RHS is a symmetric function in $\{\mathbf{x}\}$ and in $\{\mathbf{y}\}$.

Degree Naive power counting shows that the RHS is a degree $2L$ trigonometric polynomial in any rapidity variable x. However, taking the limit $x \to \infty$, for real crossing parameter, one can show explicitly that degree $2L$ terms cancel, while there are no degree $2L-1$ terms, so that the RHS is a degree $2L-2$ trigonometric polynomial.

Recursion relations The RHS satisfies the same recursion relations as LHS, as will be shown below.

Initial condition By direct calculation, the RHS for $L=1$ reduces to the weight of the $c+$ vertex.

Recursions from upper left corner. Due to the boundary conditions, the only vertices that are allowed at the upper left corner are $A(-x_1+y_1)$, $X(-x_1+y_1)$ and $C(-x_1+y_1)$.

Setting $-x_1+y_1+1=0$, we obtain $A(-1)=X(-1)=0$, which freezes all vertices on the top row and first column and leads to the recursion relation

$$Z^{1\times 1}_{L\times L}\left(\{\mathbf{x}\},\{\mathbf{y}\}|x_1=y_1+1\right) = [1][2]\left(\prod_{j=2}^{L} B(-x_1+y_{j-1})B(-x_j+y_1)\right) Z^{1\times 1}_{(L-1)\times(L-1)}\left(\{\mathbf{x}\}_1,\{\mathbf{y}\}_1\right) \tag{12}$$

Given the symmetry in vertical rapidities, we get the same relations for $x_i = y_j+1$, for all i and all j, so we have L recursion relations for each rapidity variable.

Recursions from upper right corner. Due to the boundary conditions, the only vertices that are allowed at the upper right corner are $B(-x_1+y_L)$, $Y(-x_1+y_L)$ and $C(-x_1+y_L)$.

Setting $-x_1+y_1=0$, we obtain $B(0)=Y(0)=0$, and only $C(0)$ survives at the corner, freezing all the vertices on the top row and last column. The remaining $(L-1)\times(L-1)$ lattice has once again DW boundary conditions, and we obtain the recursion relation

$$Z^{1\times 1}_{L\times L}\left(\{\mathbf{x}\},\{\mathbf{y}\}|x_1=y_L\right) = [1][2]\left(\prod_{j=2}^{L} A(-x_1+y_j)A(-x_j+y_L)\right) Z^{1\times 1}_{(L-1)\times(L-1)}\left(\{\mathbf{x}\}_1,\{\mathbf{y}\}_L\right) \tag{13}$$

Given the symmetry in vertical rapidities, a similar relation, $x_i=y_j$, can be written for all i and j, and we have L recursion relations for each rapidity variable. Thus we have altogether $2L$ recursion relations for each variable, which are sufficient to completely determine the partition function as a trigonometric polynomial of degree $2L-2$ in that variable, just as Izergin's proof.

As the RHS of equation 10 satisfies the $2L$ required recursion relations and is a polynomial of degree $2L-2$, in every rapidity variable, and given that the initial condition is satisfied, we conclude that it coincides with the LHS of equation 10.

Comments. The above Izergin type proof does not extend to higher spin models, beyond spin-1. The reason is that, for $k > 2$, the degree of the polynomials that we need to determine is higher than the number of available recursion relations[6].

6. The homogeneous Limit

Taking the homogeneous limit of the spin-$k/2$ partition function, following the footsteps of [2], is straightforward. For convenience, we re-write $M^{k/2\times k/2}_{kL\times kL,(i,j)}$ as

$$M^{k/2\times k/2}_{kL\times kL,(i,j)} = \begin{pmatrix} \phi(-x_i+y_j) & \phi(-x_i+y_j+1) & \dots & \phi(-x_i+y_j+k-1) \\ \phi(-x_i+y_j-1) & \phi(-x_i+y_j) & & \vdots \\ \vdots & & \ddots & \\ \phi(-x_i+y_j-k+1) & \dots & & \phi(-x_i+y_j) \end{pmatrix}$$

where $\phi(x) = 1/([x][x+1])$. Let $x_1 = x$ and consider the limit $x_2 \to x$. The first block row $M^{k/2\times k/2}_{kL\times kL,1j}$ remains unchanged apart from replacing x_1 with x, while $M^{k/2\times k/2}_{kL\times kL,2j}$ becomes

$$M^{k/2\times k/2}_{kL\times kL,(2,j)} = \begin{pmatrix} \phi(-x_2+y_j) & \phi(-x_2+y_j+1) & \dots & \phi(-x_2+y_j+k-1) \\ \phi(-x_2+y_j-1) & \phi(-x_2+y_j) & & \vdots \\ \vdots & & \ddots & \\ \phi(-x_2+y_j-k+1) & \dots & & \phi(-x_2+y_j) \end{pmatrix}$$

Taylor expanding each term as $x_2 \to x$, to first order, and subtracting the first block row from the second

$$(x_2-x)^k \begin{pmatrix} \phi'(-x+y_j) & \phi'(-x+y_j+1) & \dots & \phi'(-x+y_j+k-1) \\ \phi'(-x+y_j-1) & \phi'(-x+y_j) & & \vdots \\ \vdots & & \ddots & \\ \phi'(-x+y_j-k+1) & \dots & & \phi'(-x+y_j) \end{pmatrix}$$

where $\phi^{(n)}(x)$ represents the n^{th} derivative of ϕ with respect to its argument. Note that the $[-x_2+x]^k$ term in the denominator of equation 4 cancels exactly with the overall factor $(x_2-x)^k$ as $x_2 \to x$.

Taylor expanding as $x_i \to x$, up to n-th order, and successively eliminating terms by subtracting multiples of previous block rows and taking out common factors, $M^{k/2\times k/2}_{kL\times kL,ij}$ becomes

[6] This assumes that we are willing to consider only Lagrange interpolation in determining the polynomials under consideration. One can consider more elaborate interpolations, such as Hermite interpolation, but then things become very complicated.

$$(x_i - x)^{k(i-1)} \Big((i-1)! \Big)^{-k} \times$$

$$\left(\begin{array}{llll} \phi^{(i-1)}(-x+y_j) & \phi^{(i-1)}(-x+y_j+1) & \dots & \phi^{(i-1)}(-x+y_j+k-1) \\ \phi^{(i-1)}(-x+y_j-1) & \phi^{(i-1)}(-x+y_j) & & \vdots \\ \vdots & & \ddots & \\ \phi^{(i-1)}(-x+y_j-k+1) & \dots & & \phi^{(i-1)}(-x+y_j) \end{array} \right)$$

As before, the denominator in equation 4 contributes a factor of $[-x_i + x]^{k(i-1)}$ which cancels with the above coefficient as $x_i \to x$, leaving a factor of $\Big((i-1)! \Big)^{-k}$, and semi-homogeneous spin-$k/2$ $L \times L$ partition function becomes

$$\frac{\left(\prod_{j=1}^{L} \prod_{p=0}^{k} [-x + y_j + p]_k \right)^L det \left(M_{kL \times kL}^{k/2 \times k/2} \right)}{\prod_{i=1}^{k-1} \Big(-[i]^2 \Big)^{(k-i)(L^2-L)/2} \prod_{i=1}^{L-1} \Big(i! \Big)^k \prod_{1 \le i < j \le L} \prod_{p=1}^{k} [-y_i + y_j + p]_k} \tag{14}$$

With $M_{kL \times kL, ij}^{k/2 \times k/2}$ given by the above matrix. Equation 14 is the partition function for a lattice where the horizontal rapidities are homogeneous but the vertical rapidities are still distinct. Applying similar arguments to the vertical rapidities, and combining results, we obtain the following expression for the homogeneous $L \times L$ spin-$k/2$ partition function

$$\mathcal{Z}_{L \times L}^{k/2 \times k/2} = \frac{\left(\prod_{p=0}^{k} [-x + y + p]_k \right)^{L^2} det \left(M_{kL \times kL}^{k/2 \times k/2} \right)}{\prod_{i=1}^{k-1} \Big(-[i]^2 \Big)^{(k-i)(L^2-L)} \prod_{i=1}^{L-1} \Big(i! \Big)^{2k}} \tag{15}$$

where $M_{L \times L, ij}^{k/2 \times k/2}$ is

$$\left(\begin{array}{llll} \phi^{(i+j-2)}(u) & \phi^{(i+j-2)}(u+1) & \dots & \phi^{(i+j-2)}(u+k-1) \\ \phi^{(i+j-2)}(u-1) & \phi^{(i+j-2)}(u) & & \vdots \\ \vdots & & \ddots & \\ \phi^{(i+j-2)}(u-k+1) & \dots & & \phi^{(i+j-2)}(u) \end{array} \right)$$

Remarks on combinatorics. Do the higher spin determinants lead to interesting combinatorics, as in the spin-1/2 case? It is not difficult to show that there is a simple bijection between spin-$k/2$ DW configurations and extended alternating sign matrices ASM's with entries in $\{0, \pm 1, \cdots, \pm k\}$, and conditions that naturally extend those of the usual ASM's [11].

However, one can also easily check that, unlike the spin-1/2 case, for $k \ge 2$ there is no choice of crossing parameter and rapidity variables such that all weights are equal (even up to phases). This rules out 1-enumerations (but not weighted enumerations). This conclusion is substantiated by direct numerical enumerations (for $k = 2$), that lead to numbers that cannot be expressed as simple products [11]. We hope to return to these issues in a separate publication.

Appendix: Technical details

We start from the partition function of the spin-1/2 model on a $kL \times kL$ lattice

$$Z^{1/2\times 1/2}_{kL\times kL} = \frac{\prod\limits_{1\le i,j\le kL} [-x_i+y_j+1][-x_i+y_j]}{\prod\limits_{1\le i<j\le kL} [-x_i+x_j] \prod\limits_{1\le j<i\le kL} [-y_i+y_j]} det \left(M^{1/2\times 1/2}_{kL\times kL} \right) \tag{16}$$

To obtain the fused partition function, we proceed in two steps. Firstly, we set the rapidities to suitable values, then we normalize the result. Setting the rapidities to $\{\{\mathbf{x_1}|k\}, \{\mathbf{x_2}|k\}, \cdots, \{\mathbf{x_L}|k\}\}$, and $\{\{\mathbf{y_1}|k\}, \{\mathbf{y_2}|k\}, \cdots, \{\mathbf{y_L}|k\}\}$, and using $u_{ij} = -x_i + y_j$, and $v_{ij} = -x_i + x_j$, we obtain

$$\begin{aligned} &\prod_{1\le i,j\le kL} [u_{ij}] \to \\ &\qquad \prod_{1\le i,j\le L} [u_{ij}-k+1][u_{ij}-k+2]^2 \cdots [u_{ij}]^k \cdots [u_{ij}+k-2]^2[u_{ij}+k-1] \end{aligned} \tag{17}$$

$$\begin{aligned} &\prod_{1\le j<i\le kL} [v_{ij}] \to \\ &\qquad \prod_{1\le j<i\le L} [v_{ij}-k+1][v_{ij}-k+2]^2 \cdots [v_{ij}]^k \cdots [v_{ij}+k-2]^2[v_{ij}+k-1] \\ &\qquad\qquad \times \left(\prod_{p=1}^{k-1} [-p]_p \right)^L \end{aligned} \tag{18}$$

where the last factor in equation 18 comes from diagonal terms such as $[-x_2+x_1] \to [-(x_1+1)+x_1] = [-1]$. Using equations 17 and 18, the RHS of equation 16 becomes

$$\frac{\prod\limits_{1\le i,j\le L} \left(\prod\limits_{p=1}^{k} [-x_i+y_j+p]_{k+1} \right) \left(\prod\limits_{p=0}^{k-1} [-x_i+y_j+p]_{k-1} \right) det \left(M^{k/2\times k/2}_{kL\times kL} \right)}{\left(\prod\limits_{p=1}^{k-1} [-p]_p \right)^{2L} \left(\prod\limits_{1\le j<i\le L} \prod\limits_{p=0}^{k-1} [-x_i+x_j+p]_k \right) \left(\prod\limits_{1\le i<j\le L} \prod\limits_{p=0}^{k-1} [-y_i+y_j+p]_k \right)} \tag{19}$$

To normalize, we divide by a factor so that the weight of the $c+$ vertex in the spin-$k/2$ model is $[k]_k$, and multiply by a factor that accounts for the change in symmetry structure of the determinant. Together, these are

$$\frac{\left(\prod\limits_{p=1}^{k-1} [-p]_p \right)^{2L}}{\left(\prod\limits_{1\le i,j\le L} \prod\limits_{p=0}^{k-1} [-x_i+y_j+p]_{k-1} \right)}$$

and we end up with the general result in equation 4.

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